# Math 560

Implicit Differentiation

§3.6

1. Use implicit ddifferentiation to find $\dfrac{dy}{dx}$ for each curve.

   (a) $x^3 + y^3 = 9$

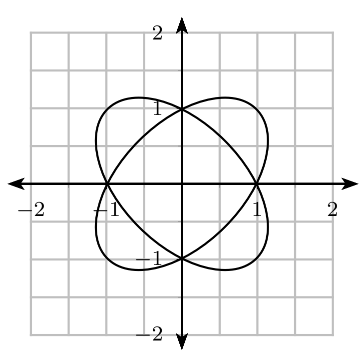
**Solution:**

$$\begin{aligned} 3x^2 + 3y^2y' &= 0 \\ y' &= \frac{-3x^2}{3y^2} \\ &= -\frac{x^2}{y^2} \end{aligned}$$

   (b) $y^2 - x^2 = x^2y^2$

**Solution:**

$$\begin{aligned} 2yy' - 2x &= x^2(2yy') + y^2(2x) \\ 2yy' - 2x^2yy' &= 2xy^2 + 2x \\ y' &= \frac{2xy^2 + 2x}{2y - 2x^2y} \\ &= \frac{xy^2 + x}{y - x^2y} \end{aligned}$$




2. The following problems refer the the implicitly defined functions $x^2+xy+y^2 = 1$ and $x^2-xy+y^2 = 1$ whose graphs are shown above.

   (a) Determine $\dfrac{dy}{dx}$ for each curve.

**Solution:** To differentiate $x^2 + xy + y^2 = 1$ we have

$$\begin{aligned} 2x + xy' + y + 2yy' &= 0 \\ y' &= \frac{-2x - y}{x + 2y} \end{aligned}$$

Similarly, the derivative of $x^2 - xy + y^2 = 1$ is $y' = \frac{y-2x}{2y-x}$.

(b) Decide which curve is which. (Hint: One way is to compare the derivatives at a common point, such as $(0, 1)$.)

**Solution:** $x^2 - xy + y^2 = 1$ is the ellipse whose major axis is rotated $\frac{\pi}{4}$ from standard postion.

(c) Determine all the points at which the tangent line is horizontal.

**Solution:** For $x^2 + xy + y^2 = 1$, the tangent line to the ellipse is horizontal when $-2x - y = 0$ or $-2x = y$. To find the $x-$coordinate we need to plug this into the original, $x^2 + xy + y^2 = 1$. So,

$$\begin{aligned} x^2 + x(-2x) + (-2x)^2 &= 1 \\ x^2 - 2x^2 + 4x^2 &= 1 \\ 3x^2 &= 1 \\ x &= \pm\frac{1}{\sqrt{3}} \end{aligned}$$

To get the $y-$coordinate, we can plug both of these value into the numerator of the derivative. So, for $x = \frac{1}{\sqrt{3}}$ we have, $y = -2\left(\frac{1}{\sqrt{3}}\right) = \frac{-2}{\sqrt{3}}$. And for $x = -\frac{1}{\sqrt{3}}$ we have, $y = -2\left(-\frac{1}{\sqrt{3}}\right) = \frac{2}{\sqrt{3}}$. Therefore, the points where the tangent line are horizontal are $\left(-\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$ and $\left(\frac{1}{\sqrt{3}}, \frac{-2}{\sqrt{3}}\right)$

We could go through this process again for the other ellipse, $x^2 - xy + y^2 = 1$, but let's exploit the fact that we have symmetry.

So, the points where the tangent is horizontal to the ellipse $x^2 - xy + y^2 = 1$ are $\left(-\frac{1}{\sqrt{3}}, \frac{-2}{\sqrt{3}}\right)$ and $\left(\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$

(d) Determine all the points at which the tangent line is vertical.

**Solution:** For $x^2 + xy + y^2 = 1$, the tangent line to the ellipse is horizontal when $x + 2y = 0$ or $x = -2y$. To find the $y-$coordinate we need to plug this into the original, $x^2 + xy + y^2 = 1$. So,

$$\begin{aligned} (-2y)^2 + (-2y)y + y^2 &= 1 \\ 4y^2 - 2y^2 + y^2 &= 1 \\ 3y^2 &= 1 \\ y &= \pm\frac{1}{\sqrt{3}} \end{aligned}$$

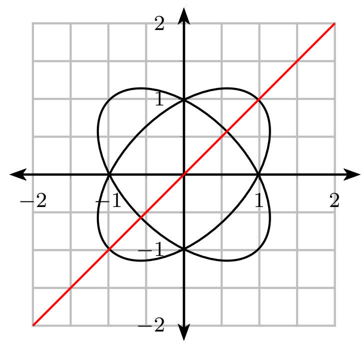
To get the $x-$coordinate, we can plug both of these value into the numerator of the derivative because we want the $x$ that makes the numerator zero. So, for $y = \frac{1}{\sqrt{3}}$ we have, $x = -2\left(\frac{1}{\sqrt{3}}\right) = \frac{-2}{\sqrt{3}}$. And for $y = -\frac{1}{\sqrt{3}}$ we have, $x = -2\left(-\frac{1}{\sqrt{3}}\right) = \frac{2}{\sqrt{3}}$.

We could go through this process again for the other ellipse, $x^2 - xy + y^2 = 1$, but let's exploit the fact that we have symmetry.

So, the points where the tangent is horizontal to the ellipse $x^2 - xy + y^2 = 1$ are $\left(\frac{-2}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$ and $\left(\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

(e) Draw the line $y = x$ on the axis above. Use derivatives to show that this line intersects both curves perpendicularly.

**Solution:**




For $x^2 - xy + y^2 = 1$, The line $y = x$ intersects the ellipse at $(1, 1)$ and $(-1, -1)$. Plugging these points into the derivative give $y' = -1$.

For $x^2 + xy + y^2 = 1$. We need to find the point of intersection. So, substituting $y = x$, we have

$$\begin{aligned} x^2 + x \cdot x + x^2 &= 1 \\ 3x^2 &= 1 \\ x &= \pm\frac{1}{\sqrt{3}} \end{aligned}$$

So, the points of intersection are $\left(\frac{1}{\sqrt{3}}, \frac{-2}{\sqrt{3}}\right)$ and $\left(\frac{-1}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$.

Then the derivative at these points is $-1$.