Math 560

Differentiablility and Continuity

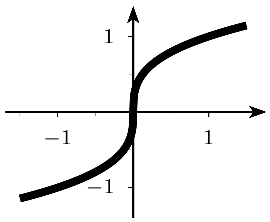
§*2.8*

In questions 1-3, (a) use a graph to explain why the derivative does or does not exist at the given point and (b) use the definition of the derivative to explain why the derivative does or does not exist at the given point.

1. Consider $f(x) = x^{\frac{1}{3}}$ at $x = 0$.

Solution:

Using a graph, show why $f'(0)$ does not exist.



From the graph it appears that the tangent line is vertical at $x = 0$.

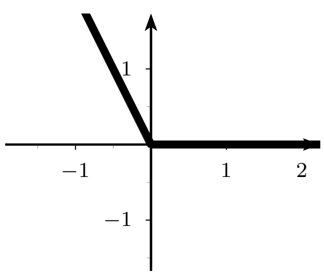
Using the definition of the derivative, show why $f'(0)$ does not exist.

$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{x^{\frac{1}{3}}-0}{x-0} = \lim_{x\to 0} \frac{1}{x^{\frac{2}{3}}} = \infty$. So no derivative at 0.

2. Consider $f(x) = |x| - x$ at $x = 0$.

Solution:

Using a graph, show why $f'(0)$ does not exist.



From the graph it appears to have a sharp corner at $x = 0$.

Using the definition of the derivative, show why $f'(0)$ does not exist.

$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{(-x-x)-0}{x-0} = \lim_{x\to 0^-} -2 = -2$.

$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{(x-x)-0}{x-0} = \lim_{x\to 0^+} 0 = 0$.

Since the limit from the left is different than the limit from the right, $f'(0)$ does not exist.

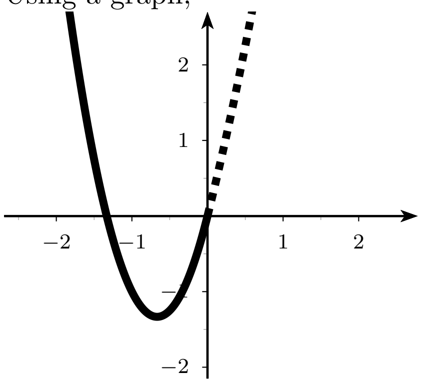
3. Consider

$$f(x) = \begin{cases} 3x^2 + 4x & \text{when } x < 0 \\ x^2 + 4x & \text{when } x \geq 0 \end{cases}$$

at $x = 0$.

Solution:

Using a graph,



From the graph it appears that there is a smooth transition at $x = 0$ so a derivative might be possible. Confirm with algebra below.

Using the definition of the derivative, show why $f'(0)$ does exist.
$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{3x^2+4x-0}{x-0} = \lim_{x\to 0^-} 3x + 4 = 4.$
$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{x^2+4x-0}{x-0} = \lim_{x\to 0^+} x + 4 = 4.$
Since the limit from the left is equal to the limit from the right, $f'(0) = 4$.

4. Consider the following function where a and b are constants.

$$f(x) = \begin{cases} 3 - x & \text{when } 1 > x \\ ax^2 + bx & \text{when } x \geq 1 \end{cases}$$

(a) Determine an equation in terms of a and b such that f is continuous for all x.

Solution: In order for f to be continuous for all x, it must be continuous at $x = 1$. Therefore, (1) $\lim_{x\to 1} f(x)$ must exist, (2) $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$, and (3) $\lim_{x\to 1} f(x) = f(1)$. Hence,

$$\begin{aligned} \lim_{x\to 1^-} 3 - x &= \lim_{x\to 1^+} ax^2 + bx \\ 2 &= a + b \end{aligned}$$

(b) Determine values for a and b such that f is continuous and differentiable for all x.

Solution: In part (a) we determined that if $a + b = 2$ then f is continuous. So, for f to be differentiable, both the left- and right-hand derivatives must be the same. Hence, the left-hand derivative is

$$\lim_{x\to 1^-} \frac{3 - (x + h) - (3 - x)}{h} = -1$$

The right-hand derivative is

$$\begin{aligned} \lim_{x\to 1^+} \frac{a(x+h)^2 + b(x+h) - (ax^2 + bx)}{h} &= \lim_{x\to 1^+} \frac{ax^2 + 2axh + ah^2 + bx + bh - bx}{h} \\ &= \lim_{x\to 1^+} \frac{2axh + bh}{h} \\ &= 2ax + b \end{aligned}$$

Thus, when $x = 1$ both the left- and right-hand derivatives must be equal. So we have $-1 = 2a + b$. Solving the system $a + b = 2$ and $-1 = 2a + b$ for a and b yields $a = -3$ and $b = 5$.